

Explicit computations of low lying eigenfunctions for the quantum trigonometric Calogero-Sutherland model related to the exceptional algebra E_7

J. Fernández Núñez, W. García Fuertes, A.M. Perelomov*

Departamento de Física, Facultad de Ciencias, Universidad de Oviedo, E-33007 Oviedo, Spain

Abstract

In a previous paper [1] we have studied the characters and Clebsch-Gordan series for the exceptional Lie algebra E_7 by relating them to the quantum trigonometric Calogero-Sutherland Hamiltonian with coupling constant $\kappa = 1$. Now we extend that approach to the case of general κ .

1 Introduction

The Calogero-Sutherland models [2, 3] related to the root systems of the simple Lie algebras [4, 5, 6] have been deeply investigated during the last two decades. Originally introduced on purely theoretical grounds, this class of models have however found a number of relevant applications in such diverse fields as condensed matter physics, supersymmetric Yang-Mills theory or black-hole physics. On the mathematical side, an interesting feature of the quantum version of this kind of models is that their energy eigenfunctions provide a natural generalization of several types of hypergeometric functions to the multivariable case. For the potential $v(q) = \kappa(\kappa - 1)\sin^{-2}(q)$ and special values of the coupling constant, these eigenfunctions are related to some orthogonal functional systems of particular interest in the theory of Lie algebras and symmetric spaces: for $\kappa = 1$ we obtain the characters of the irreducible representations of the algebra, while for $\kappa = 0$ the corresponding monomial symmetric functions arise; other values of κ lead to zonal spherical functions in symmetric spaces associated to the Lie algebra: in particular, for E_7 , $\kappa = \frac{1}{2}$ gives these functions for the symmetric space EV^* [7, 6]. The Calogero-Sutherland Hamiltonian appears in this way as a natural unified tool for the computation of all these objects.

The Calogero-Sutherland Hamiltonian associated to the root system of a simple Lie algebra can be written as a second-order differential operator whose variables are the characters of the fundamental representations of the algebra. As it was shown in the papers [8, 9, 10], and later in [11, 12, 13, 14, 15, 16, 17], this approach gives the possibility of developing some systematic procedures to solve the Schrödinger equation and determine important properties of the eigenfunctions, such as recurrence relations or generating functions for some subsets of them. The approach has been used for classical algebras of A_n and D_n type, for the exceptional algebra E_6 , and recently also for E_7 for the special value of the coupling constant for which the eigenfunctions are proportional to the characters of the irreducible representations of the algebra. The aim of this paper is to show how to generalize the treatment given in [1] to arbitrary values of the coupling constant and to extend some of the particular results found there to the general case.

*On leave of absence from the Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia. Current E-mail address: perelomo@dftuz.unizar.es

2 The Calogero-Sutherland Hamiltonian for E_7 in Weyl-invariant variables

The trigonometric Calogero-Sutherland model related to the root system \mathcal{R} of a simply-laced Lie algebra of rank r is the quantum system in an Euclidean space \mathbf{R}^r defined by the standard Hamiltonian operator

$$H = \frac{1}{2} \sum_{j=1}^r p_j^2 + \sum_{\alpha \in \mathcal{R}^+} \kappa_\alpha (\kappa_\alpha - 1) \sin^{-2}(\alpha, q), \quad (1)$$

where $q = (q_j)$ is a cartesian coordinate system and $p_j = -i \partial_{q_j}$; \mathcal{R}^+ is the set of the positive roots of the algebra, and κ is the coupling constant. The (non-normalized) ground state wave function is

$$\Psi_0^\kappa(q) = \prod_{\alpha \in \mathcal{R}^+} \sin^\kappa(\alpha, q), \quad (2)$$

while the excited states are indexed by the highest weights $\mu = \sum m_i l_i \in P^+$ (P^+ is the cone of dominant weights) of the irreducible representations of the algebra, that is, by the r -tuple of non-negative integers $\mathbf{m} = (m_1, \dots, m_r)$. Looking for solutions $\Psi_{\mathbf{m}}^\kappa$ of the Schrödinger equation in the form

$$\Psi_{\mathbf{m}}^\kappa(q) = \Psi_0^\kappa(q) \Phi_{\mathbf{m}}^\kappa(q), \quad (3)$$

we are led to the eigenvalue problem

$$\Delta^\kappa \Phi_{\mathbf{m}}^\kappa = \varepsilon_{\mathbf{m}}(\kappa) \Phi_{\mathbf{m}}^\kappa, \quad (4)$$

where Δ^κ is the linear differential operator

$$\Delta^\kappa = -\frac{1}{2} \sum_{j=1}^r \partial_{q_j}^2 - \kappa \sum_{\alpha \in \mathcal{R}^+} \cot(\alpha, q) (\alpha, \nabla_q). \quad (5)$$

Due to the Weyl symmetry of the Hamiltonian, to solve the eigenvalue problem (4) it is convenient to express the operator Δ^κ in a set of independent W -invariant variables such as $z_k = \chi_{l_k}(q)$, the characters of the irreducible representations of the algebra. The operator Δ^κ in the z -variables has the structure:

$$\Delta^\kappa = \sum_{j,k} a_{jk}(z) \partial_{z_j} \partial_{z_k} + \sum_j [b_j^0(z) + \kappa b_j^1(z)] \partial_{z_j}, \quad (6)$$

but to fix the coefficients by direct change of variables is very cumbersome. As explained in [1], a different procedure, based on the computation of the quadratic Clebsch-Gordan series and the second order characters of E_7 , is possible. In [1], we applied this procedure to compute $a_{jk}(z)$ and $b_j^0(z) + b_j^1(z)$, thus finding the operator Δ^κ for the special case $\kappa = 1$. The remaining task is to compute $b_j^0(z)$ and $b_j^1(z)$ separately.

To accomplish that task a new piece of information is required: we need to know all the first order symmetric monomials for E_7 given as a function of the z -variables. To obtain them, we will rely on the expansions of the fundamental characters of E_7 in terms of monomial functions computed by Loris and Sasaki in [18]. In the notation of [1], these expansions are

$$\begin{aligned} z_1 &= M_{\lambda_1} + 7, \\ z_2 &= M_{\lambda_2} + 6M_{\lambda_7}, \\ z_3 &= M_{\lambda_3} + 5M_{\lambda_6} + 22M_{\lambda_1} + 77, \\ z_4 &= M_{\lambda_4} + 4M_{\lambda_1+\lambda_6} + 15M_{\lambda_2+\lambda_7} + 15M_{2\lambda_1} + 45M_{2\lambda_7} + 50M_{\lambda_3} + 145M_{\lambda_6} + 390M_{\lambda_1} + 980, \\ z_5 &= M_{\lambda_5} + 5M_{\lambda_1+\lambda_7} + 21M_{\lambda_2} + 71M_{\lambda_7}, \\ z_6 &= M_{\lambda_6} + 6M_{\lambda_1} + 27, \\ z_7 &= M_{\lambda_7}. \end{aligned}$$

To invert these formulas to compute the fundamental monomial functions, we have to proceed in increasing order of the height of the dominant weights associated to the characters. Once a first-order monomial function is known,

we compute the corresponding $b_k^0(z)$ and $b_k^1(z)$ following the procedure described in [1]. This makes it possible to use the part of the operator Δ^κ already known in each step to compute the second order monomial functions in advance, i.e. before they are needed to obtain the next fundamental monomial function. With this strategy, it is easy to find

$$\begin{aligned}
M_{\lambda_1} &= z_1 - 7, \\
M_{\lambda_2} &= z_2 - 6z_7, \\
M_{\lambda_3} &= z_3 - 5z_6 + 8z_1 + 2, \\
M_{\lambda_4} &= z_4 - 4z_1z_6 + 9z_2z_7 + 9z_1^2 + 9z_7^2 - 14z_3 - 39z_6 - 22z_1 - 18, \\
M_{\lambda_5} &= z_5 - 5z_1z_7 + 14z_2 + 15z_7, \\
M_{\lambda_6} &= z_6 - 6z_1 + 15, \\
M_{\lambda_7} &= z_7,
\end{aligned}$$

and, therefore,

$$\begin{aligned}
b_1^0(z) + \kappa b_1^1(z) &= -28 + 4z_1 + \kappa(28 + 68z_1) \\
b_2^0(z) + \kappa b_2^1(z) &= 7z_2 - 24z_7 + \kappa(98z_2 + 24z_7) \\
b_3^0(z) + \kappa b_3^1(z) &= 8 - 56z_1 + 12z_3 - 20z_6 + \kappa(-8 + 56z_1 + 132z_3 + 20z_6) \\
b_4^0(z) + \kappa b_4^1(z) &= -72 + 72z_1 - 24z_1^2 + 24z_3 + 24z_4 - 16z_6 - 16z_1z_6 - 24z_2z_7 + 36z_7^2 + \\
&\quad \kappa(72 - 72z_1 + 24z_1^2 - 24z_3 + 192z_4 + 16z_6 + 16z_1z_6 + 24z_2z_7 - 36z_7^2) \\
b_5^0(z) + \kappa b_5^1(z) &= -28z_2 + 15z_5 - 4z_7 - 20z_1z_7 + \kappa(28z_2 + 150z_5 + 4z_7 + 20z_1z_7) \\
b_6^0(z) + \kappa b_6^1(z) &= -48 - 24z_1 + 8z_6 + \kappa(48 + 24z_1 + 104z_6) \\
b_7^0(z) + \kappa b_7^1(z) &= 3z_7 + 54\kappa z_7.
\end{aligned}$$

This completes the computation of Δ^κ . In the rest of the paper we present some results obtained through the use of this operator.

3 Some explicit results on the low lying eigenfunctions of the systems

In this Section, we present some results on the first and second order polynomials and generalized quadratic Clebsch-Gordan series. Because some formulas are too long, we give the complete results, in a form suitable for use in Mathematica or Maple, in the adjoint files `results31.txt` – `results35.txt`, which are accessible through the “source” format of this document.

3.1 Second order monomial symmetric functions

Once we know Δ^κ , we can compute its eigenfunctions by means of the iterative algorithms given in [1]. In particular, we can obtain the monomial symmetric functions for E_7 by simply taken $\kappa = 0$ in these algorithms. We present here the list of the second order monomial functions obtained in that way.

$$\begin{aligned}
M_{2000000} &= z_1^2 - 2z_3 - 2z_1 - 7 \\
M_{1100000} &= z_1z_2 - 5z_5 + 3z_1z_7 - 23z_7 \\
M_{0200000} &= z_2^2 - 2z_4 - 2z_2z_7 - 2z_1^2 - 6z_7^2 + 4z_3 + 14z_6 + 4z_1 + 12 \\
M_{1010000} &= z_1z_3 - 3z_4 - z_1z_6 + 6z_2z_7 - 3z_1^2 - 9z_7^2 - 9z_3 + 4z_6 + 20z_1 + 32 \\
M_{0110000} &= z_2z_3 - 4z_1z_5 + 5z_2z_6 + 4z_1^2z_7 - 4z_3z_7 - 17z_1z_2 - 16z_6z_7 + 41z_5 + 13z_1z_7 + 12z_2 + 7z_7 \\
M_{0020000} &= z_3^2 - 2z_1z_4 + 2z_2z_5 - 2z_3z_6 - 7z_6^2 + 12z_5z_7 - 2z_1^3 + 4z_1z_3 + 2z_1z_7^2 - 10z_4 - 2z_1z_6 + 10z_1^2 \\
&\quad + 10z_7^2 - 16z_3 - 22z_6 - 16z_1 - 8 \\
M_{1001000} &= z_1z_4 - 4z_2z_5 - 4z_1^2z_6 + 10z_3z_6 + 9z_1z_2z_7 + 14z_6^2 - 34z_5z_7 + 9z_1^3 - 21z_2^2 - 39z_1z_3 - 21z_1z_7^2 \\
&\quad + 66z_4 + 54z_1z_6 + 23z_2z_7 + 36z_1^2 - 22z_7^2 - 54z_3 - 24z_6 - 56z_1 - 24 \\
M_{0101000} &= z_2z_4 - 3z_3z_5 + 2z_1z_2z_6 - 2z_2^2z_7 + 5z_1z_3z_7 + 5z_5z_6 - 14z_4z_7 - 19z_1z_6z_7 - 12z_1^2z_2 + 15z_2z_3
\end{aligned}$$

$$\begin{aligned}
& + 17z_2z_7^2 + 25z_1z_5 + 19z_1^2z_7 + 42z_7^3 + 5z_2z_6 - 29z_3z_7 - 27z_1z_2 - 133z_6z_7 + 131z_5 + 10z_1z_7 \\
& + 40z_2 - 43z_7 \\
M_{0011000} & = z_3z_4 - 3z_1z_2z_5 + 5z_2^2z_6 + 2z_1z_3z_6 + 5z_5^2 - 7z_4z_6 + 5z_1^2z_2z_7 - 10z_1z_6^2 - 14z_2z_3z_7 + 10z_1z_5z_7 \\
& - 12z_1^2z_3 - 3z_1^2z_7^2 - 5z_1z_2^2 + 24z_3^2 + 5z_1z_4 - 2z_2z_6z_7 + 6z_1^2z_6 + z_2z_5 + 11z_3z_7^2 + 10z_3z_6 \\
& + 15z_6z_7^2 - 28z_1z_2z_7 - 24z_6^2 + 40z_2^2 + 4z_5z_7 + 21z_1^3 - 15z_1z_3 + 19z_1z_7^2 - 16z_4 - 54z_1z_6 \\
& - 48z_1^2 + 17z_2z_7 - 5z_7^2 + 7z_3 + 31z_6 - 16z_1 - 22 \\
M_{0002000} & = z_4^2 - 2z_2z_3z_5 + 2z_3^2z_6 + 2z_1z_5^2 + 2z_1z_2^2z_6 - 2z_2^3z_7 - 4z_1z_4z_6 - 2z_1z_2z_3z_7 - 2z_2z_5z_6 + 6z_2z_4z_7 \\
& - 8z_1^2z_6^2 + 12z_3z_6^2 + 12z_1^2z_5z_7 - 20z_3z_5z_7 - 6z_1^2z_4 + 2z_1z_2z_6z_7 + 2z_6^3 + 2z_2^2z_3 + 2z_1^3z_7^2 \\
& - 4z_5z_6z_7 + 9z_2^2z_7^2 - 4z_1z_3z_7^2 + 12z_3z_4 - 4z_1z_2z_5 - 16z_2^2z_6 - 14z_4z_7^2 - 4z_1^3z_6 + 2z_5^2 + 8z_1z_3z_6 \\
& + 2z_1^2z_2z_7 + 30z_4z_6 + 9z_1^4 - 18z_2z_7^3 + 12z_1z_6^2 - 6z_2z_3z_7 - 16z_1z_5z_7 - 36z_1^2z_3 + 4z_1z_2^2 - 8z_1^2z_7^2 \\
& + 26z_3^2 + 32z_2z_6z_7 + 16z_3z_7^2 + 32z_1z_4 - 28z_2z_5 + 14z_1^2z_6 + 9z_7^4 - 22z_6z_7^2 - 16z_1^3 - 5z_6^2 \\
& - 8z_1z_2z_7 + 32z_1z_3 - 16z_2^2 + 52z_5z_7 - 48z_4 - 10z_1z_7^2 + 36z_2z_7 + 42z_7^2 + 20z_1^2 - 8z_3 - 88z_6 \\
& - 8z_1 - 60 \\
M_{1000100} & = z_1z_5 - 5z_2z_6 - 5z_1^2z_7 + 15z_3z_7 + 9z_1z_2 + 19z_6z_7 - 54z_5 - 29z_1z_7 + 30z_2 + 56z_7 \\
M_{0100100} & = z_2z_5 - 4z_3z_6 + 4z_6^2 + 5z_1z_2z_7 - 7z_2^2 + 4z_1z_3 - 4z_5z_7 - 10z_4 - 16z_1z_7^2 + 4z_1z_6 + 29z_2z_7 \\
& + 54z_7^2 + 6z_1^2 - 12z_3 - 90z_6 - 76z_1 + 4 \\
M_{0010100} & = z_3z_5 - 4z_1z_2z_6 + 9z_2^2z_7 + 5z_1z_3z_7 + 5z_5z_6 - 12z_4z_7 - 11z_1z_6z_7 + 4z_1^2z_2 - 25z_2z_3 + 16z_1z_5 \\
& - 4z_1^2z_7 - 5z_2z_6 + 19z_3z_7 - 2z_1z_2 + 31z_6z_7 - 46z_5 - 26z_1z_7 - 40z_2 + 74z_7 \\
M_{0001100} & = z_4z_5 - 3z_2z_3z_6 + 2z_1z_5z_6 + 5z_3^2z_7 + 5z_2z_6^2 + 5z_1z_2^2z_7 - 7z_1z_4z_7 - 7z_2^3 - 12z_1z_2z_3 - 14z_2z_5z_7 \\
& + 27z_2z_4 - 10z_1^2z_6z_7 + 10z_3z_6z_7 - 2z_1z_2z_7^2 - 3z_6^2z_7 + 21z_1^2z_5 - 17z_3z_5 + 10z_1z_2z_6 + 11z_5z_7^2 \\
& + 5z_1^3z_7 - 8z_5z_6 + 15z_1z_7^3 + 6z_2^2z_7 - 5z_1z_3z_7 - 5z_1^2z_2 - 17z_4z_7 + 16z_2z_3 - 26z_1z_6z_7 - 16z_1z_5 \\
& - 45z_2z_7^2 - z_1^2z_7 + 45z_2z_6 + 5z_7^3 + 31z_3z_7 - 26z_1z_2 + 14z_6z_7 + 42z_5 - 26z_1z_7 + 110z_2 - 26z_7 \\
M_{0000200} & = z_5^2 - 2z_4z_6 + 2z_2z_3z_7 - 2z_3^2 - 2z_1z_5z_7 - 2z_1z_2^2 - 7z_1^2z_7^2 + 4z_1z_4 + 12z_3z_7^2 + 2z_2z_5 + 14z_1^2z_6 \\
& - 24z_3z_6 + 2z_1z_2z_7 + 2z_6z_7^2 + 4z_1^3 - 4z_6^2 + 2z_5z_7 + 14z_2^2 + 10z_1z_7^2 - 8z_1z_3 - 28z_4 - 24z_1z_6 \\
& - 26z_2z_7 + 17z_7^2 - 20z_1^2 + 32z_3 - 8z_6 + 32z_1 - 32 \\
M_{1000010} & = z_1z_6 - 6z_2z_7 + 9z_7^2 - 6z_1^2 + 21z_3 + 6z_6 - 27z_1 + 27 \\
M_{0100010} & = z_2z_6 - 5z_3z_7 + 3z_6z_7 + 9z_1z_2 - 7z_5 - 19z_1z_7 + 15z_2 + 7z_7 \\
M_{0010010} & = z_3z_6 - 5z_6^2 - 5z_1z_2z_7 + 14z_2^2 + 9z_1z_3 + 15z_5z_7 - 27z_4 + 19z_1z_7^2 - 49z_1z_6 - 14z_2z_7 - 27z_1^2 \\
& - 4z_7^2 + 42z_3 + 79z_6 + 12z_1 - 42 \\
M_{0001010} & = z_4z_6 - 4z_2z_3z_7 - 4z_1z_6^2 + 9z_3^2 + 10z_1z_5z_7 + 9z_2z_6z_7 + 9z_1z_2^2 + 14z_1^2z_7^2 - 18z_1z_4 - 34z_3z_7^2 \\
& - 33z_2z_5 - 39z_1^2z_6 + 72z_3z_6 - 5z_1z_2z_7 - 21z_6z_7^2 - 18z_1^3 + 34z_6^2 + 22z_5z_7 - 14z_2^2 + 18z_1z_7^2 \\
& + 48z_1z_3 - 7z_4 - 16z_2z_7 - 61z_7^2 - 27z_1^2 + 18z_3 + 151z_6 + 150z_1 + 26 \\
M_{0000110} & = z_5z_6 - 3z_4z_7 - z_1z_6z_7 + 5z_2z_3 + 6z_2z_7^2 - 2z_1z_5 - 11z_2z_6 - 9z_7^3 - 9z_1^2z_7 + 9z_3z_7 + 7z_1z_2 \\
& + 25z_6z_7 - 16z_5 + 26z_1z_7 - 60z_2 + 16z_7 \\
M_{0000020} & = z_6^2 - 2z_5z_7 + 2z_4 - 6z_1^2 - 2z_7^2 + 12z_3 + 4z_6 + 12z_1 + 17 \\
M_{1000001} & = z_1z_7 - 7z_2 + 8z_7 \\
M_{0100001} & = z_2z_7 - 6z_3 - 6z_7^2 + 14z_6 + 16z_1 - 28 \\
M_{0010001} & = z_3z_7 - 5z_6z_7 - 6z_1z_2 + 20z_5 + 24z_1z_7 - 49z_2 - 12z_7 \\
M_{0001001} & = z_4z_7 - 4z_1z_6z_7 - 5z_2z_3 + 9z_2z_7^2 + 14z_1z_5 - 5z_2z_6 + 9z_7^3 + 19z_1^2z_7 - 44z_3z_7 - 14z_1z_2 \\
& - 53z_6z_7 + 54z_5 + 42z_1z_7 - 40z_2 - 144z_7 \\
M_{0000101} & = z_5z_7 - 4z_4 - 5z_1z_7^2 + 8z_1z_6 + 11z_2z_7 + 12z_1^2 - 3z_7^2 - 42z_3 - 18z_6 - 4 \\
M_{0000011} & = z_6z_7 - 3z_5 - z_1z_7 + 7z_2 - 11z_7 \\
M_{0000002} & = z_7^2 - 2z_6 - 2
\end{aligned}$$

3.2 Expansion of second order characters in monomial functions

As the name suggests, the orthogonal system of monomial symmetric functions is the simplest one among the different classes of symmetric polynomials associated to the Lie algebra E_7 : each monomial symmetric function is nothing but the sum of all the monomials associated to one orbit of the Weyl group on the weight lattice. Now, we can easily expand other polynomials associated to the root system of E_7 in the basis of the monomial symmetric functions. In fact, the method is the same which we have described in [1] for the computation of Clebsch-Gordan series. In particular, the coefficients in the decomposition of characters in monomial symmetric functions are interesting in that they give the multiplicities of the weights in the corresponding irreducible representations. As an example, we present such decomposition for all the second order characters.

$$\begin{aligned}
\chi_{2000000} &= M_{2000000} + M_{0010000} + 4M_{0000010} + 17M_{1000000} + 63M_{0000000} \\
\chi_{1100000} &= M_{1100000} + 4M_{0000100} + 16M_{1000001} + 56M_{0100000} + 171M_{0000001} \\
\chi_{0200000} &= M_{0200000} + M_{0001000} + 3M_{1000010} + 11M_{0100001} + 10M_{2000000} + 36M_{0000002} + 34M_{0010000} \\
&\quad + 96M_{0000010} + 248M_{1000000} + 603M_{0000000} \\
\chi_{1010000} &= M_{1010000} + 2M_{0001000} + 8M_{1000010} + 24M_{0100001} + 32M_{2000000} + 64M_{0000002} + 78M_{0010000} \\
&\quad + 208M_{0000010} + 544M_{1000000} + 1344M_{0000000} \\
\chi_{0110000} &= M_{0110000} + 3M_{1000100} + 10M_{0100010} + 10M_{2000001} + 30M_{0010001} + 90M_{1100000} + 80M_{0000011} \\
&\quad + 231M_{0000100} + 570M_{1000001} + 1344M_{0100000} + 3024M_{0000001} \\
\chi_{0020000} &= M_{0020000} + M_{1001000} + 2M_{0100100} + 3M_{2000010} + 7M_{0010010} + 19M_{1100001} + 20M_{0000020} \\
&\quad + 46M_{0000101} + 10M_{3000000} + 49M_{0200000} + 56M_{1010000} + 104M_{1000002} + 125M_{0001000} \\
&\quad + 291M_{1000010} + 682M_{2000000} + 638M_{0100001} + 1338M_{0000002} + 1402M_{0010000} + 2908M_{0000010} \\
&\quad + 5938M_{1000000} + 11844M_{0000000} \\
\chi_{1001000} &= M_{1001000} + 3M_{0100100} + 4M_{2000010} + 10M_{0010010} + 30M_{1100001} + 25M_{0000020} + 75M_{0000101} \\
&\quad + 15M_{3000000} + 84M_{0200000} + 90M_{1010000} + 180M_{1000002} + 213M_{0001000} + 507M_{1000010} \\
&\quad + 1149M_{0100001} + 1185M_{2000000} + 2484M_{0000002} + 2565M_{0010000} + 5439M_{0000010} \\
&\quad + 11265M_{1000000} + 22680M_{0000000} \\
\chi_{0101000} &= M_{0101000} + 2M_{0010100} + 6M_{1100010} + 20M_{0200001} + 15M_{1010001} + 15M_{0000110} + 42M_{0001001} \\
&\quad + 96M_{1000011} + 40M_{2100000} + 114M_{0110000} + 220M_{0100002} + 256M_{1000100} + 565M_{2000001} \\
&\quad + 480M_{0000003} + 575M_{0100010} + 1240M_{0010001} + 2624M_{1100000} + 2580M_{0000011} \\
&\quad + 5340M_{0000100} + 10589M_{1000001} + 20524M_{0100000} + 38864M_{0000001} \\
\chi_{0011000} &= M_{0011000} + 2M_{1100100} + 5M_{0200010} + 6M_{1010010} + 5M_{0000200} + 14M_{0001010} + 15M_{2100001} \\
&\quad + 33M_{1000020} + 37M_{0110001} + 83M_{1000101} + 40M_{2010000} + 180M_{2000002} + 94M_{1200000} \\
&\quad + 100M_{0020000} + 215M_{1001000} + 180M_{0100011} + 467M_{2000010} + 456M_{0100100} + 375M_{0010002} \\
&\quad + 958M_{0010010} + 750M_{0000012} + 1964M_{1100001} + 1920M_{0000020} + 3963M_{0200000} + 3850M_{0000101} \\
&\quad + 1010M_{3000000} + 4005M_{1010000} + 7374M_{1000002} + 7700M_{0001000} + 14642M_{1000010} \\
&\quad + 27546M_{2000000} + 27263M_{0100001} + 49698M_{0000002} + 50206M_{0010000} + 90408M_{0000010} \\
&\quad + 160642M_{1000000} + 281268M_{0000000} \\
\chi_{0002000} &= M_{0002000} + M_{0110100} + 2M_{0020010} + 2M_{1000200} + 2M_{1200010} + 5M_{0300001} + 5M_{1001010} \\
&\quad + 11M_{1110001} + 11M_{0100110} + 27M_{0101001} + 12M_{2000020} + 25M_{2200000} + 23M_{0010020} \\
&\quad + 23M_{2000101} + 54M_{0010101} + 25M_{1020000} + 52M_{2001000} + 109M_{1100011} + 45M_{0000030} \\
&\quad + 64M_{0210000} + 45M_{3000002} + 210M_{0000111} + 225M_{0200002} + 210M_{1010002} + 129M_{0011000} \\
&\quad + 258M_{1100100} + 520M_{0200010} + 408M_{0001002} + 750M_{1000012} + 105M_{3000010} + 499M_{0000200} \\
&\quad + 501M_{1010010} + 960M_{2100001} + 968M_{0001010} + 215M_{4000000} + 1365M_{0100003} + 1787M_{1000020} \\
&\quad + 1854M_{0110001} + 3376M_{1000101} + 1830M_{2010000} + 3524M_{1200000} + 6055M_{2000002} \\
&\quad + 3525M_{0020000} + 6085M_{0100011} + 10760M_{0010002} + 6350M_{1001000} + 11358M_{0100100}
\end{aligned}$$

$$\begin{aligned}
& + 11320M_{2000010} + 2440M_{0000004} + 18700M_{0000012} + 20031M_{3000000} + 19977M_{0010010} \\
& + 34569M_{0000020} + 34769M_{1100001} + 60006M_{1010000} + 60004M_{0200000} + 59439M_{0000101} \\
& + 101592M_{0001000} + 100299M_{1000002} + 170142M_{1000010} + 281804M_{0100001} + 461353M_{0000002} \\
& + 282527M_{2000000} + 462702M_{0010000} + 750988M_{0000010} + 1208053M_{1000000} + 1925763M_{0000000} \\
\chi_{1000100} & = M_{1000100} + 4M_{0100010} + 5M_{2000001} + 15M_{0010001} + 50M_{1100000} + 44M_{0000011} + 139M_{0000100} \\
& + 365M_{1000001} + 910M_{0100000} + 2145M_{0000001} \\
\chi_{0100100} & = M_{0100100} + 3M_{0010010} + 10M_{0000020} + 10M_{1100001} + 35M_{0200000} + 29M_{1010000} + 30M_{0000101} \\
& + 88M_{0001000} + 80M_{1000002} + 223M_{1000010} + 545M_{0100001} + 1260M_{0000002} + 538M_{2000000} \\
& + 1262M_{0010000} + 2800M_{0000010} + 5976M_{1000000} + 12341M_{0000000} \\
\chi_{0010100} & = M_{0010100} + 3M_{1100010} + 9M_{0200001} + 10M_{1010001} + 10M_{0000110} + 28M_{0001001} + 72M_{1000011} \\
& + 29M_{2100000} + 169M_{0100002} + 79M_{0110000} + 196M_{1000100} + 374M_{0000003} + 464M_{2000001} \\
& + 458M_{0100010} + 1029M_{0010001} + 2258M_{1100000} + 2198M_{0000011} + 4708M_{0000100} + 9574M_{1000001} \\
& + 18998M_{0100000} + 36774M_{0000001} \\
\chi_{0001100} & = M_{0001100} + 2M_{0110010} + 6M_{1000110} + 5M_{0020001} + 15M_{0100020} + 5M_{1200001} + 14M_{1001001} \\
& + 14M_{0300000} + 34M_{1110000} + 37M_{0100101} + 91M_{0101000} + 33M_{2000011} + 83M_{0010011} + 180M_{1100002} \\
& + 180M_{0000021} + 78M_{2000100} + 203M_{0010100} + 437M_{1100010} + 375M_{0000102} + 170M_{3000001} \\
& + 914M_{0000110} + 750M_{1000003} + 929M_{0200001} + 905M_{1010001} + 1834M_{2100000} + 1858M_{0001001} \\
& + 3723M_{0110000} + 3635M_{1000011} + 7156M_{1000100} + 6949M_{0100002} + 13480M_{2000001} + 13524M_{0100010} \\
& + 12954M_{0000003} + 25015M_{0010001} + 45599M_{1100000} + 45368M_{0000011} + 81502M_{0000100} \\
& + 143470M_{1000001} + 249025M_{0100000} + 426280M_{0000001} \\
\chi_{0000200} & = M_{0000200} + M_{0001010} + 2M_{0110001} + 3M_{1000020} + 5M_{0020000} + 7M_{1000101} + 19M_{0100011} \\
& + 5M_{1200000} + 20M_{2000002} + 16M_{1001000} + 46M_{0010002} + 46M_{0100100} + 41M_{2000010} + 110M_{0010010} \\
& + 250M_{1100001} + 104M_{0000012} + 94M_{3000000} + 254M_{0000020} + 549M_{0000101} + 560M_{0200000} \\
& + 1150M_{1000002} + 539M_{1010000} + 1159M_{0001000} + 2362M_{1000010} + 4700M_{0100001} + 9126M_{0000002} \\
& + 4678M_{2000000} + 9103M_{0010000} + 17256M_{0000010} + 32022M_{1000000} + 58324M_{0000000} \\
\chi_{1000010} & = M_{1000010} + 5M_{0100001} + 20M_{0000002} + 6M_{2000000} + 21M_{0010000} + 70M_{0000010} + 212M_{1000000} \\
& + 588M_{0000000} \\
\chi_{0100010} & = M_{0100010} + 4M_{0010001} + 16M_{0000011} + 15M_{1100000} + 51M_{0000100} + 149M_{1000001} + 399M_{0100000} \\
& + 999M_{0000001} \\
\chi_{0010010} & = M_{0010010} + 5M_{0000020} + 4M_{1100001} + 14M_{0200000} + 15M_{1010000} + 15M_{0000101} + 47M_{0001000} \\
& + 44M_{1000002} + 133M_{1000010} + 343M_{0100001} + 350M_{2000000} + 828M_{0000002} + 845M_{0010000} \\
& + 1957M_{0000010} + 4366M_{1000000} + 9387M_{0000000} \\
\chi_{0001010} & = M_{0001010} + 3M_{0110001} + 4M_{1000020} + 9M_{0020000} + 10M_{1000101} + 30M_{0100011} + 9M_{1200000} \\
& + 25M_{2000002} + 27M_{1001000} + 75M_{0010002} + 78M_{0100100} + 69M_{2000010} + 193M_{0010010} + 449M_{1100001} \\
& + 180M_{0000012} + 165M_{3000000} + 460M_{0000020} + 1014M_{0000101} + 1008M_{0200000} + 2169M_{1000002} \\
& + 999M_{1010000} + 2184M_{0001000} + 4549M_{1000010} + 9198M_{0100001} + 18063M_{0000002} + 9189M_{2000000} \\
& + 18114M_{0010000} + 34807M_{0000010} + 65475M_{1000000} + 120771M_{0000000} \\
\chi_{0000110} & = M_{0000110} + 2M_{0001001} + 8M_{1000011} + 5M_{0110000} + 24M_{0100002} + 19M_{1000100} + 59M_{0100010} \\
& + 64M_{0000003} + 54M_{2000001} + 154M_{0010001} + 374M_{1100000} + 384M_{0000011} + 879M_{0000100} \\
& + 1958M_{1000001} + 4193M_{0100000} + 8694M_{0000001} \\
\chi_{0000020} & = M_{0000020} + M_{0000101} + 2M_{0001000} + 4M_{1000002} + 9M_{1000010} + 29M_{0100001} + 30M_{2000000} \\
& + 84M_{0000002} + 80M_{0010000} + 209M_{0000010} + 510M_{1000000} + 1197M_{0000000} \\
\chi_{1000001} & = M_{1000001} + 6M_{0100000} + 27M_{0000001}
\end{aligned}$$

$$\begin{aligned}
\chi_{0100001} &= M_{0100001} + 5M_{0010000} + 6M_{0000002} + 22M_{0000010} + 75M_{1000000} + 225M_{0000000} \\
\chi_{0010001} &= M_{0010001} + 5M_{0000011} + 5M_{1100000} + 20M_{0000100} + 66M_{1000001} + 196M_{0100000} + 531M_{0000001} \\
\chi_{0001001} &= M_{0001001} + 4M_{1000011} + 4M_{0110000} + 15M_{0100002} + 14M_{1000100} + 45M_{0100010} \\
&\quad + 45M_{0000003} + 40M_{2000001} + 125M_{0010001} + 319M_{1100000} + 325M_{0000011} + 784M_{0000100} \\
&\quad + 1809M_{1000001} + 4004M_{0100000} + 8529M_{0000001} \\
\chi_{0000101} &= M_{0000101} + 3M_{0001000} + 5M_{1000002} + 13M_{1000010} + 45M_{0100001} + 39M_{2000000} + 135M_{0000002} \\
&\quad + 129M_{0010000} + 351M_{0000010} + 879M_{1000000} + 2079M_{0000000} \\
\chi_{0000011} &= M_{0000011} + 2M_{0000100} + 10M_{1000001} + 35M_{0100000} + 111M_{0000001} \\
\chi_{0000002} &= M_{0000002} + M_{0000010} + 5M_{1000000} + 21M_{0000000}
\end{aligned}$$

3.3 First order polynomials

The iterative methods given in [1] allow us to solve the Schrödinger equation (4) for general κ . The eigenfunctions are polynomials. In this and the next subsection, we present a partial list of such polynomials of first and second order.

$$\begin{aligned}
P_{1000000}^\kappa(z) &= z_1 + \frac{7(-1+\kappa)}{1+17\kappa} \\
P_{0100000}^\kappa(z) &= z_2 + \frac{6(-1+\kappa)z_7}{1+11\kappa} \\
P_{0010000}^\kappa(z) &= z_3 + \frac{5(-1+\kappa)z_6}{1+7\kappa} + \frac{8(-1+\kappa)(-1+8\kappa)z_1}{(1+7\kappa)(1+8\kappa)} + \frac{2(-1+\kappa)(-1-159\kappa+136\kappa^2)}{(1+7\kappa)(1+8\kappa)(1+11\kappa)} \\
P_{0001000}^\kappa(z) &= z_4 + \frac{4(-1+\kappa)z_1z_6}{1+5\kappa} + \frac{3(-1+\kappa)(-3+5\kappa)z_2z_7}{(1+5\kappa)^2} + \frac{-9(-1+\kappa)(1+22\kappa+5\kappa^2)z_7^2}{(1+5\kappa)^2(1+7\kappa)} \\
&\quad + \frac{9(-1+\kappa)(-1+3\kappa)z_1^2}{(1+5\kappa)(1+7\kappa)} + \frac{-2(-1+\kappa)(-7+11\kappa+385\kappa^2+175\kappa^3)z_3}{(1+5\kappa)^3(1+7\kappa)} \\
&\quad + \frac{(-1+\kappa)(78+1255\kappa+3653\kappa^2-6295\kappa^3+3325\kappa^4)z_6}{(1+5\kappa)^3(1+7\kappa)(2+11\kappa)} \\
&\quad + \frac{-2(-1+\kappa)(-22-755\kappa-3477\kappa^2+11255\kappa^3+175\kappa^4)z_1}{(1+5\kappa)^3(1+7\kappa)(2+11\kappa)} \\
&\quad + \frac{2(-1+\kappa)(18+365\kappa+8123\kappa^2-2045\kappa^3+2275\kappa^4)}{(1+5\kappa)^3(1+7\kappa)(2+11\kappa)} \\
P_{0000100}^\kappa(z) &= z_5 + \frac{5(-1+\kappa)z_1z_7}{1+7\kappa} + \frac{7(-1+\kappa)(-4+7\kappa)z_2}{(1+7\kappa)(2+13\kappa)} + \frac{5(-1+\kappa)(-6-137\kappa+56\kappa^2)z_7}{(1+7\kappa)(1+8\kappa)(2+13\kappa)} \\
P_{0000010}^\kappa(z) &= z_6 + \frac{6(-1+\kappa)z_1}{1+9\kappa} + \frac{15(-1+\kappa)(-1+5\kappa)}{(1+9\kappa)(1+13\kappa)} \\
P_{0000001}^\kappa(z) &= z_7
\end{aligned}$$

3.4 Second order polynomials

$$\begin{aligned}
P_{2000000}^\kappa(z) &= z_1^2 + \frac{-2z_3}{1+\kappa} + \frac{-10\kappa z_6}{(1+\kappa)(1+4\kappa)} + \frac{2(-3-6\kappa-119\kappa^2+28\kappa^3)z_1}{(1+\kappa)(1+4\kappa)(3+17\kappa)} \\
&\quad + \frac{-42-459\kappa-290\kappa^2-3205\kappa^3+196\kappa^4}{(1+\kappa)(1+4\kappa)(2+17\kappa)(3+17\kappa)} \\
P_{1100000}^\kappa(z) &= z_1z_2 + \frac{-5z_5}{1+4\kappa} + \frac{(6-95\kappa+24\kappa^2)z_1z_7}{(1+4\kappa)(2+11\kappa)} + \frac{28(-1+\kappa)\kappa(-26+11\kappa)z_2}{(1+4\kappa)(2+11\kappa)(3+17\kappa)} \\
&\quad + \frac{(-1+\kappa)(138+365\kappa-9979\kappa^2+1176\kappa^3)z_7}{(1+4\kappa)(1+7\kappa)(2+11\kappa)(3+17\kappa)}
\end{aligned}$$

$$\begin{aligned}
P_{0200000}^\kappa(z) &= z_2^2 + \frac{-2z_4}{1+\kappa} + \frac{-8\kappa z_1 z_6}{(1+\kappa)(1+3\kappa)} + \frac{6(-1+\kappa)(1-\kappa+6\kappa^2)z_2 z_7}{(1+\kappa)(1+3\kappa)(3+11\kappa)} + \frac{-2(1+23\kappa^2)z_1^2}{(1+\kappa)(1+3\kappa)(1+5\kappa)} \\
&+ \frac{18(-1+\kappa)(2+13\kappa-7\kappa^2+6\kappa^3)z_7^2}{(1+\kappa)(1+3\kappa)(2+11\kappa)(3+11\kappa)} + \frac{4(3+23\kappa+141\kappa^2+493\kappa^3+180\kappa^4)z_3}{(1+\kappa)(1+3\kappa)(1+4\kappa)(1+5\kappa)(3+11\kappa)} \\
&+ \frac{-2(-42-537\kappa-2397\kappa^2-6715\kappa^3-14529\kappa^4+2380\kappa^5)z_6}{(1+\kappa)(1+3\kappa)(1+4\kappa)(1+5\kappa)(2+11\kappa)(3+11\kappa)} \\
&+ \frac{-4(-6-99\kappa-205\kappa^2-1021\kappa^3-12161\kappa^4+2572\kappa^5)z_1}{(1+\kappa)(1+3\kappa)(1+4\kappa)(1+5\kappa)(2+11\kappa)(3+11\kappa)} \\
&+ \frac{-4(-1+\kappa)(18+325\kappa+2143\kappa^2+2067\kappa^3-14045\kappa^4+22012\kappa^5)}{(1+\kappa)(1+3\kappa)(1+4\kappa)(1+5\kappa)(1+7\kappa)(2+11\kappa)(3+11\kappa)} \\
P_{1010000}^\kappa(z) &= z_1 z_3 + \frac{-3z_4}{1+2\kappa} + \frac{(-2-35\kappa+10\kappa^2)z_1 z_6}{(1+2\kappa)(2+7\kappa)} + \frac{-6(-1+\kappa)(2+15\kappa)z_2 z_7}{(1+2\kappa)(1+4\kappa)(2+7\kappa)} \\
&+ \frac{(-18-47\kappa-704\kappa^2+256\kappa^3)z_1^2}{(1+2\kappa)(2+7\kappa)(3+16\kappa)} + \frac{9(-2+17\kappa)z_7^2}{(1+2\kappa)(1+4\kappa)(2+7\kappa)} \\
&+ \frac{(-216-2054\kappa+1429\kappa^2+33210\kappa^3+15224\kappa^4+6272\kappa^5)z_3}{(1+2\kappa)(1+4\kappa)(2+7\kappa)(3+16\kappa)(4+17\kappa)} \\
&+ \frac{2(-1+\kappa)(-48+1190\kappa+6697\kappa^2-9260\kappa^3+2240\kappa^4)z_6}{(1+2\kappa)(1+4\kappa)(2+7\kappa)(3+16\kappa)(4+17\kappa)} \\
&+ \frac{12(-1+\kappa)(-80-100\kappa+7708\kappa^2+27189\kappa^3-30716\kappa^4+12736\kappa^5)z_1}{(1+2\kappa)(1+4\kappa)(2+7\kappa)(2+11\kappa)(3+16\kappa)(4+17\kappa)} \\
&+ \frac{4(-1+\kappa)(-384-6676\kappa-12672\kappa^2+67253\kappa^3-75612\kappa^4+7616\kappa^5)}{(1+2\kappa)(1+4\kappa)(2+7\kappa)(2+11\kappa)(3+16\kappa)(4+17\kappa)} \\
P_{0110000}^\kappa(z) &= z_2 z_3 + \frac{-4z_1 z_5}{1+3\kappa} + \frac{5(-1+\kappa)(-2+3\kappa)z_2 z_6}{(1+3\kappa)(2+7\kappa)} + \frac{-4(-1+7\kappa)z_1^2 z_7}{(1+3\kappa)(1+5\kappa)} \\
&+ \frac{6(-4+51\kappa+311\kappa^2+41\kappa^3+105\kappa^4)z_3 z_7}{(1+3\kappa)(1+5\kappa)(2+7\kappa)(3+11\kappa)} \\
&+ \frac{2(-1+\kappa)(51+397\kappa-6\kappa^2-2992\kappa^3+2640\kappa^4)z_1 z_2}{(1+3\kappa)(1+4\kappa)(1+5\kappa)(2+7\kappa)(3+11\kappa)} \\
&+ \frac{6(-16+17\kappa+673\kappa^2-245\kappa^3+75\kappa^4)z_6 z_7}{(1+3\kappa)(1+5\kappa)(2+7\kappa)(3+11\kappa)} \\
&+ \frac{2(-1+\kappa)(-123-821\kappa+1018\kappa^2+10196\kappa^3+1280\kappa^4)z_5}{(1+3\kappa)(1+4\kappa)^2(1+5\kappa)(2+7\kappa)(3+11\kappa)} \\
&+ \frac{6(13-156\kappa-2853\kappa^2-2376\kappa^3+27380\kappa^4-11328\kappa^5+1920\kappa^6)z_1 z_7}{(1+3\kappa)(1+4\kappa)^2(1+5\kappa)(2+7\kappa)(3+11\kappa)} \\
&+ \frac{6(-1+\kappa)(-12-595\kappa-2610\kappa^2+7325\kappa^3-2248\kappa^4+1360\kappa^5)z_2}{(1+3\kappa)(1+4\kappa)^2(1+5\kappa)(2+7\kappa)(3+11\kappa)} \\
&+ \frac{2(-1+\kappa)(-42-3713\kappa-46855\kappa^2-49890\kappa^3+620062\kappa^4-178192\kappa^5+24480\kappa^6)z_7}{(1+3\kappa)(1+4\kappa)^2(1+5\kappa)(2+7\kappa)(2+11\kappa)(3+11\kappa)} \\
P_{1000100}^\kappa(z) &= z_1 z_5 + \frac{-5z_2 z_6}{1+4\kappa} + \frac{5(-1+\kappa)z_1^2 z_7}{1+7\kappa} + \frac{-15(-1+\kappa)(1+6\kappa)z_3 z_7}{(1+4\kappa)^2(1+7\kappa)} \\
&+ \frac{(-1+\kappa)(-27-320\kappa+112\kappa^2)z_1 z_2}{(1+4\kappa)^2(3+13\kappa)} + \frac{-(-1+\kappa)(19+309\kappa+1182\kappa^2)z_6 z_7}{(1+4\kappa)^2(1+5\kappa)(1+7\kappa)} \\
&+ \frac{(-1+\kappa)(648+8119\kappa+26227\kappa^2+12296\kappa^3+7280\kappa^4)z_5}{(1+4\kappa)^2(1+5\kappa)(3+13\kappa)(4+17\kappa)} \\
&+ \frac{2(-174-4381\kappa-20915\kappa^2+61085\kappa^3+363609\kappa^4-239624\kappa^5+42000\kappa^6)z_1 z_7}{(1+4\kappa)^2(1+5\kappa)(1+7\kappa)(3+13\kappa)(4+17\kappa)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{10(36 + 449\kappa - 4256\kappa^2 - 63640\kappa^3 - 162486\kappa^4 + 81791\kappa^5 - 107534\kappa^6 + 13720\kappa^7)z_2}{(1 + 4\kappa)^2(1 + 5\kappa)^2(1 + 7\kappa)(3 + 13\kappa)(4 + 17\kappa)} \\
& + \frac{(-1 + \kappa)(-672 - 17905\kappa - 81245\kappa^2 + 497285\kappa^3 + 2671037\kappa^4 - 1392420\kappa^5 + 98000\kappa^6)z_7}{(1 + 4\kappa)^2(1 + 5\kappa)^2(1 + 7\kappa)(3 + 13\kappa)(4 + 17\kappa)} \\
P_{1000010}^\kappa(z) & = z_1 z_6 + \frac{-6z_2 z_7}{1 + 5\kappa} + \frac{-9(-1 + 7\kappa)z_7^2}{(1 + 5\kappa)(1 + 8\kappa)} + \frac{6(-1 + \kappa)z_1^2}{1 + 9\kappa} + \frac{-3(-1 + \kappa)(7 + 55\kappa)z_3}{(1 + 5\kappa)^2(1 + 9\kappa)} \\
& + \frac{(18 + 1213\kappa + 15375\kappa^2 + 51579\kappa^3 - 15985\kappa^4 + 12600\kappa^5)z_6}{(1 + 5\kappa)^2(1 + 8\kappa)(1 + 9\kappa)(3 + 17\kappa)} \\
& + \frac{3(-54 - 775\kappa + 4734\kappa^2 + 107248\kappa^3 + 344272\kappa^4 - 252825\kappa^5 + 121400\kappa^6)z_1}{(1 + 5\kappa)^2(1 + 8\kappa)(1 + 9\kappa)(2 + 13\kappa)(3 + 17\kappa)} \\
& + \frac{3(54 - 245\kappa - 9444\kappa^2 + 22702\kappa^3 + 391238\kappa^4 - 115305\kappa^5 + 35000\kappa^6)}{(1 + 5\kappa)^2(1 + 8\kappa)(1 + 9\kappa)(2 + 13\kappa)(3 + 17\kappa)} \\
P_{0100010}^\kappa(z) & = z_2 z_6 + \frac{-5z_3 z_7}{1 + 4\kappa} + \frac{(6 - 95\kappa + 24\kappa^2)z_6 z_7}{(1 + 4\kappa)(2 + 11\kappa)} + \frac{6(-1 + \kappa)(-3 + 4\kappa)z_1 z_2}{(1 + 4\kappa)(2 + 9\kappa)} \\
& + \frac{2(-42 + 643\kappa + 4519\kappa^2 + 600\kappa^3)z_5}{(1 + 4\kappa)(2 + 9\kappa)(2 + 11\kappa)(3 + 13\kappa)} \\
& + \frac{2(-1 + \kappa)(114 + 113\kappa - 11559\kappa^2 - 46218\kappa^3 + 4680\kappa^4)z_1 z_7}{(1 + 4\kappa)(1 + 5\kappa)(2 + 9\kappa)(2 + 11\kappa)(3 + 13\kappa)} \\
& + \frac{(180 - 3864\kappa - 20509\kappa^2 + 80848\kappa^3 - 4515\kappa^4 + 16500\kappa^5)z_2}{(1 + 4\kappa)(1 + 5\kappa)(2 + 9\kappa)(2 + 11\kappa)(3 + 13\kappa)} \\
& + \frac{2(-1 + \kappa)(-42 + 2347\kappa + 38355\kappa^2 - 27714\kappa^3 + 4500\kappa^4)z_7}{(1 + 4\kappa)(1 + 5\kappa)(2 + 9\kappa)(2 + 11\kappa)(3 + 13\kappa)} \\
P_{0000020}^\kappa(z) & = z_6^2 + \frac{-2z_5 z_7}{1 + \kappa} + \frac{-2(-1 + \kappa)z_4}{(1 + \kappa)(1 + 2\kappa)} + \frac{-10\kappa z_1 z_7^2}{(1 + \kappa)(1 + 4\kappa)} + \frac{4\kappa(13 + 43\kappa + 10\kappa^2 + 24\kappa^3)z_1 z_6}{3(1 + \kappa)(1 + 2\kappa)(1 + 3\kappa)(1 + 4\kappa)} \\
& + \frac{-2(-1 + \kappa)\kappa(7 + 36\kappa)z_2 z_7}{(1 + \kappa)(1 + 2\kappa)(1 + 3\kappa)(1 + 4\kappa)} + \frac{2(-1 + \kappa)(6 + 47\kappa + 59\kappa^2 - 88\kappa^3 + 48\kappa^4)z_1^2}{(1 + \kappa)(1 + 2\kappa)(1 + 3\kappa)(1 + 4\kappa)(2 + 9\kappa)} \\
& + \frac{-2(-1 + \kappa)(-1 - 8\kappa + 30\kappa^2)z_7^2}{(1 + \kappa)(1 + 2\kappa)(1 + 3\kappa)(1 + 4\kappa)} + \frac{4(6 + 29\kappa + 36\kappa^2 + 199\kappa^3 + 60\kappa^4)z_3}{(1 + \kappa)(1 + 2\kappa)(1 + 3\kappa)(1 + 4\kappa)(2 + 9\kappa)} \\
& + \frac{4(18 + 213\kappa - 140\kappa^2 - 1100\kappa^3 + 11362\kappa^4 + 2787\kappa^5 + 2700\kappa^6)z_6}{3(1 + \kappa)(1 + 2\kappa)(1 + 3\kappa)(1 + 4\kappa)(2 + 9\kappa)(3 + 13\kappa)} \\
& + \frac{4(18 + 285\kappa + 949\kappa^2 - 2675\kappa^3 - 5493\kappa^4 + 33950\kappa^5 + 1646\kappa^6 + 3000\kappa^7)z_1}{(1 + \kappa)(1 + 2\kappa)(1 + 3\kappa)(1 + 4\kappa)(1 + 5\kappa)(2 + 9\kappa)(3 + 13\kappa)} \\
& + \frac{(-1 + \kappa)(-204 - 4208\kappa - 37487\kappa^2 - 140165\kappa^3 - 24391\kappa^4 + 655613\kappa^5 - 66238\kappa^6 + 75000\kappa^7)}{(1 + \kappa)(1 + 2\kappa)(1 + 3\kappa)(1 + 4\kappa)(1 + 5\kappa)(2 + 9\kappa)(2 + 13\kappa)(3 + 13\kappa)} \\
P_{1000001}^\kappa(z) & = z_1 z_7 + \frac{-7z_2}{1 + 6\kappa} + \frac{(16 - 191\kappa + 42\kappa^2)z_7}{(1 + 6\kappa)(2 + 17\kappa)} \\
P_{0100001}^\kappa(z) & = z_2 z_7 + \frac{-6z_3}{1 + 5\kappa} + \frac{6(-1 + \kappa)z_7^2}{1 + 11\kappa} + \frac{-2(-7 - 31\kappa + 290\kappa^2)z_6}{(1 + 5\kappa)(1 + 6\kappa)(1 + 11\kappa)} \\
& + \frac{-16(-1 - 26\kappa - 119\kappa^2 + 398\kappa^3)z_1}{(1 + 5\kappa)(1 + 6\kappa)(1 + 7\kappa)(1 + 11\kappa)} + \frac{-4(-1 + \kappa)(-7 - 51\kappa + 826\kappa^2)}{(1 + 5\kappa)(1 + 6\kappa)(1 + 7\kappa)(1 + 11\kappa)} \\
P_{0010001}^\kappa(z) & = z_3 z_7 + \frac{5(-1 + \kappa)z_6 z_7}{1 + 7\kappa} + \frac{-6z_1 z_2}{1 + 5\kappa} + \frac{-10(-1 + \kappa)(4 + 25\kappa)z_5}{(1 + 5\kappa)(1 + 7\kappa)(2 + 9\kappa)} \\
& + \frac{2(-1 + \kappa)(-72 - 1183\kappa - 4188\kappa^2 + 2880\kappa^3)z_1 z_7}{(1 + 5\kappa)(1 + 7\kappa)(2 + 9\kappa)(3 + 16\kappa)} \\
& + \frac{-28(-1 + \kappa)(-21 - 260\kappa - 333\kappa^2 + 3056\kappa^3)z_2}{(1 + 5\kappa)(1 + 7\kappa)(2 + 9\kappa)(2 + 11\kappa)(3 + 16\kappa)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{2(-72 - 4430\kappa - 9270\kappa^2 + 121301\kappa^3 - 54561\kappa^4 + 12240\kappa^5)z_7}{(1+5\kappa)(1+7\kappa)(2+9\kappa)(2+11\kappa)(3+16\kappa)} \\
P_{0000101}^\kappa(z) &= z_5 z_7 + \frac{-4z_4}{1+3\kappa} + \frac{5(-1+\kappa)z_1 z_7^2}{1+7\kappa} + \frac{-8(-1-\kappa+22\kappa^2)z_1 z_6}{(1+3\kappa)(1+4\kappa)(1+7\kappa)} \\
& + \frac{(-1+\kappa)(-33-239\kappa+84\kappa^2)z_2 z_7}{(1+3\kappa)(1+4\kappa)(3+13\kappa)} + \frac{-12(-1+\kappa)(1+12\kappa)z_1^2}{(1+3\kappa)(1+4\kappa)(1+7\kappa)} \\
& + \frac{(-9+327\kappa+3089\kappa^2-1267\kappa^3+420\kappa^4)z_7^2}{(1+3\kappa)(1+4\kappa)(1+7\kappa)(3+13\kappa)} + \frac{-12(-1+\kappa)(-21-118\kappa+8\kappa^2)z_3}{(1+3\kappa)(1+4\kappa)(2+9\kappa)(3+13\kappa)} \\
& + \frac{-4(27+1050\kappa+8824\kappa^2+18278\kappa^3-11811\kappa^4+25872\kappa^5)z_6}{(1+3\kappa)(1+4\kappa)(1+5\kappa)(1+7\kappa)(2+9\kappa)(3+13\kappa)} \\
& + \frac{-48(-1+\kappa)\kappa(-89-1205\kappa-3637\kappa^2+2159\kappa^3)z_1}{(1+3\kappa)(1+4\kappa)(1+5\kappa)(1+7\kappa)(2+9\kappa)(3+13\kappa)} \\
& + \frac{-8(3-203\kappa+3439\kappa^2+24833\kappa^3-17242\kappa^4+10290\kappa^5)}{(1+3\kappa)(1+4\kappa)(1+5\kappa)(1+7\kappa)(2+9\kappa)(3+13\kappa)} \\
P_{0000011}^\kappa(z) &= z_6 z_7 + \frac{-3z_5}{1+2\kappa} + \frac{(-2-43\kappa+12\kappa^2)z_1 z_7}{(1+2\kappa)(2+9\kappa)} + \frac{-7(-1+\kappa)(2+19\kappa)z_2}{(1+2\kappa)(1+5\kappa)(2+9\kappa)} \\
& + \frac{2(-1+\kappa)(22-2\kappa-1052\kappa^2+375\kappa^3)z_7}{(1+2\kappa)(1+5\kappa)(2+9\kappa)(2+13\kappa)} \\
P_{0000002}^\kappa(z) &= z_7^2 + \frac{-2z_6}{1+\kappa} + \frac{-12\kappa z_1}{(1+\kappa)(1+5\kappa)} + \frac{-2(1+59\kappa^2)}{(1+\kappa)(1+5\kappa)(1+9\kappa)}
\end{aligned}$$

3.5 Generalized quadratic Clebsch-Gordan series

For each κ , the product of polynomials can be decomposed as a linear combination of polynomials of the same κ . The terms entering in this decomposition are exactly the same entering in the product of characters, i.e. in the corresponding Clebsch-Gordan series, while the coefficients are rational functions of κ . The method for computing these coefficients was explained in [1]. Here we give some of the quadratic Clebsch-Gordan series for general κ .

$$\begin{aligned}
P_{1000000}^\kappa \times P_{1000000}^\kappa &= P_{2000000}^\kappa + \frac{2}{1+\kappa} P_{0010000}^\kappa + \frac{10(1+3\kappa)}{(1+4\kappa)(1+7\kappa)} P_{0000010}^\kappa \\
& + \frac{24(4+103\kappa+547\kappa^2+696\kappa^3)}{(1+8\kappa)(1+9\kappa)(1+17\kappa)(3+17\kappa)} P_{1000000}^\kappa \\
& + \frac{252(1+3\kappa)(1+5\kappa)(1+8\kappa)(1+18\kappa)}{(1+11\kappa)(1+13\kappa)(1+17\kappa)^2(2+17\kappa)} \\
P_{1000000}^\kappa \times P_{0100000}^\kappa &= P_{1100000}^\kappa + \frac{5}{1+4\kappa} P_{0000100}^\kappa + \frac{32(1+2\kappa)(1+12\kappa)}{(1+7\kappa)(1+11\kappa)(2+11\kappa)} P_{1000001}^\kappa \\
& + \frac{42(1+3\kappa)(1+14\kappa)(5+101\kappa+74\kappa^2)}{(1+6\kappa)(1+11\kappa)(2+13\kappa)(1+17\kappa)(3+17\kappa)} P_{0100000}^\kappa \\
& + \frac{144(1+2\kappa)(1+3\kappa)(1+5\kappa)(1+18\kappa)}{(1+7\kappa)(1+8\kappa)(1+11\kappa)^2(2+17\kappa)} P_{0000001}^\kappa \\
P_{1000000}^\kappa \times P_{0010000}^\kappa &= P_{1010000}^\kappa + \frac{3}{1+2\kappa} P_{0001000}^\kappa + \frac{16(1+2\kappa)(1+8\kappa)}{(1+5\kappa)(1+7\kappa)(2+7\kappa)} P_{1000010}^\kappa \\
& + \frac{15(1+\kappa)(1+3\kappa)(1+11\kappa)}{(1+4\kappa)(1+5\kappa)^2(1+7\kappa)} P_{0100001}^\kappa + \frac{48(1+2\kappa)(1+4\kappa)(2+17\kappa)}{(1+7\kappa)(1+8\kappa)(1+9\kappa)(3+16\kappa)} P_{2000000}^\kappa \\
& + \frac{-24(1+11\kappa)(-5-187\kappa-2180\kappa^2-9982\kappa^3-18875\kappa^4-11395\kappa^5+1800\kappa^6)}{(1+\kappa)(1+5\kappa)^3(1+7\kappa)(1+8\kappa)(1+17\kappa)(4+17\kappa)} P_{0010000}^\kappa \\
& + \frac{240(1+\kappa)(1+2\kappa)(1+3\kappa)(1+12\kappa)(1+13\kappa)}{(1+6\kappa)(1+7\kappa)^2(1+8\kappa)(2+11\kappa)(3+17\kappa)} P_{0000010}^\kappa
\end{aligned}$$

$$\begin{aligned}
& + \frac{384(1+2\kappa)(1+3\kappa)(1+4\kappa)^2(1+5\kappa)(1+12\kappa)(1+14\kappa)(1+17\kappa)}{(1+7\kappa)^2(1+8\kappa)^2(1+9\kappa)(1+11\kappa)(2+11\kappa)(2+13\kappa)(3+17\kappa)} P_{1000000}^\kappa \\
P_{1000000}^\kappa \times P_{0000100}^\kappa &= P_{1000100}^\kappa + \frac{5}{1+4\kappa} P_{0100010}^\kappa + \frac{10(1+\kappa)(1+9\kappa)}{(1+4\kappa)^2(1+7\kappa)} P_{0010001}^\kappa \\
& + \frac{240(1+\kappa)(1+2\kappa)(1+10\kappa)}{(1+5\kappa)(2+9\kappa)(2+13\kappa)(3+13\kappa)} P_{1100000}^\kappa + \frac{32(1+2\kappa)(1+3\kappa)(1+12\kappa)}{(1+5\kappa)(1+7\kappa)^2(2+11\kappa)} P_{0000011}^\kappa \\
& + \frac{-90(1+10\kappa)(-16-562\kappa-5649\kappa^2-18716\kappa^3-18653\kappa^4+3276\kappa^5)}{(1+4\kappa)(1+7\kappa)(2+9\kappa)(2+13\kappa)(3+13\kappa)(1+17\kappa)(4+17\kappa)} P_{0000100}^\kappa \\
& + \frac{240(1+\kappa)(1+2\kappa)(1+3\kappa)(2+7\kappa)(1+10\kappa)(1+12\kappa)(2+17\kappa)}{(1+5\kappa)(1+7\kappa)^2(1+8\kappa)(2+9\kappa)(2+11\kappa)(2+13\kappa)(3+16\kappa)} P_{1000001}^\kappa \\
& + \frac{420(1+\kappa)(1+2\kappa)(1+3\kappa)(1+4\kappa)(2+7\kappa)(1+11\kappa)(1+12\kappa)(1+14\kappa)}{(1+5\kappa)^2(1+6\kappa)(1+7\kappa)(1+8\kappa)(2+11\kappa)(2+13\kappa)^2(3+17\kappa)} P_{0100000}^\kappa \\
P_{1000000}^\kappa \times P_{0000010}^\kappa &= P_{1000010}^\kappa + \frac{6}{1+5\kappa} P_{0100001}^\kappa + \frac{27(1+3\kappa)}{(1+8\kappa)(1+11\kappa)} P_{0000002}^\kappa + \frac{15(1+\kappa)(1+11\kappa)}{(1+5\kappa)^2(1+9\kappa)} P_{0010000}^\kappa \\
& + \frac{-48(1+3\kappa)(1+13\kappa)(-2-45\kappa-109\kappa^2+6\kappa^3)}{(1+\kappa)(1+6\kappa)(1+7\kappa)(1+9\kappa)(1+17\kappa)(3+17\kappa)} P_{0000010}^\kappa \\
& + \frac{120(1+\kappa)(1+3\kappa)(1+4\kappa)(1+6\kappa)(1+14\kappa)(1+17\kappa)}{(1+5\kappa)(1+7\kappa)(1+8\kappa)(1+9\kappa)^2(1+13\kappa)(2+13\kappa)} P_{1000000}^\kappa \\
P_{1000000}^\kappa \times P_{0000001}^\kappa &= P_{1000001}^\kappa + \frac{7}{1+6\kappa} P_{0100000}^\kappa + \frac{54(1+3\kappa)(1+18\kappa)}{(1+11\kappa)(1+17\kappa)(2+17\kappa)} P_{0000001}^\kappa \\
P_{0100000}^\kappa \times P_{0100000}^\kappa &= P_{0200000}^\kappa + \frac{2}{1+\kappa} P_{0001000}^\kappa + \frac{8(1+2\kappa)}{(1+3\kappa)(1+5\kappa)} P_{1000010}^\kappa \\
& + \frac{12(5+84\kappa+255\kappa^2+160\kappa^3)}{(1+5\kappa)^2(1+11\kappa)(3+11\kappa)} P_{0100001}^\kappa + \frac{32(1+2\kappa)(1+4\kappa)}{(1+5\kappa)(1+7\kappa)(1+9\kappa)} P_{2000000}^\kappa \\
& + \frac{144(1+2\kappa)(1+3\kappa)(1+5\kappa)(1+12\kappa)}{(1+7\kappa)(1+8\kappa)(1+11\kappa)^2(2+11\kappa)} P_{0000002}^\kappa \\
& + \frac{-40(1+2\kappa)^2(-1-22\kappa-59\kappa^2+10\kappa^3)}{(1+\kappa)(1+4\kappa)(1+5\kappa)^3(1+11\kappa)} P_{0010000}^\kappa \\
& + \frac{-160(1+2\kappa)^2(1+3\kappa)(1+12\kappa)(-3-64\kappa+11\kappa^2)}{(1+\kappa)(1+6\kappa)(1+7\kappa)(1+11\kappa)^2(2+11\kappa)(3+17\kappa)} P_{0000010}^\kappa \\
& + \frac{192(1+2\kappa)(1+3\kappa)(1+4\kappa)(1+14\kappa)(5+101\kappa+74\kappa^2)}{(1+7\kappa)(1+8\kappa)(1+9\kappa)(1+11\kappa)^2(2+13\kappa)(3+17\kappa)} P_{1000000}^\kappa \\
& + \frac{1152(1+2\kappa)(1+3\kappa)(1+4\kappa)(1+5\kappa)(1+6\kappa)(1+18\kappa)}{(1+7\kappa)(1+9\kappa)(1+11\kappa)^2(1+13\kappa)(1+17\kappa)(2+17\kappa)} \\
P_{0100000}^\kappa \times P_{0000010}^\kappa &= P_{0100010}^\kappa + \frac{5}{1+4\kappa} P_{0010001}^\kappa + \frac{32(1+2\kappa)(1+12\kappa)}{(1+7\kappa)(1+11\kappa)(2+11\kappa)} P_{0000011}^\kappa \\
& + \frac{30(1+\kappa)(1+10\kappa)}{(1+5\kappa)(1+9\kappa)(2+9\kappa)} P_{1100000}^\kappa \\
& + \frac{-60(1+10\kappa)(-3-56\kappa-119\kappa^2+18\kappa^3)}{(1+4\kappa)(1+9\kappa)(2+9\kappa)(1+11\kappa)(3+13\kappa)} P_{0000100}^\kappa \\
& + \frac{48(1+2\kappa)(1+3\kappa)(14+531\kappa+6042\kappa^2+23399\kappa^3+24354\kappa^4)}{(1+5\kappa)(1+7\kappa)(1+9\kappa)(2+9\kappa)(1+11\kappa)(2+11\kappa)(3+16\kappa)} P_{1000001}^\kappa \\
& + \frac{-420(1+2\kappa)(1+3\kappa)(1+4\kappa)(1+12\kappa)(1+14\kappa)(-3-64\kappa+11\kappa^2)}{(1+5\kappa)(1+6\kappa)(1+9\kappa)(1+11\kappa)(2+11\kappa)(1+13\kappa)(2+13\kappa)(3+17\kappa)} P_{0100000}^\kappa \\
& + \frac{864(1+\kappa)(1+2\kappa)(1+3\kappa)(1+4\kappa)(1+14\kappa)(1+18\kappa)}{(1+7\kappa)(1+8\kappa)(1+9\kappa)(1+11\kappa)^2(2+13\kappa)(2+17\kappa)} P_{0000001}^\kappa
\end{aligned}$$

$$\begin{aligned}
P_{0100000}^\kappa \times P_{0000001}^\kappa &= P_{0100001}^\kappa + \frac{6}{1+5\kappa} P_{0010000}^\kappa + \frac{16(1+2\kappa)(1+13\kappa)}{(1+6\kappa)(1+7\kappa)(1+11\kappa)} P_{0000010}^\kappa \\
&+ \frac{32(1+2\kappa)(1+4\kappa)(1+17\kappa)}{(1+7\kappa)(1+8\kappa)(1+9\kappa)(1+11\kappa)} P_{1000000}^\kappa \\
P_{0010000}^\kappa \times P_{0000001}^\kappa &= P_{0010001}^\kappa + \frac{6}{1+5\kappa} P_{1100000}^\kappa + \frac{20(1+\kappa)(1+10\kappa)}{(1+4\kappa)(1+7\kappa)(2+9\kappa)} P_{0000100}^\kappa \\
&+ \frac{48(1+2\kappa)(1+3\kappa)(1+12\kappa)(2+17\kappa)}{(1+7\kappa)^2(1+8\kappa)(2+11\kappa)(3+16\kappa)} P_{1000001}^\kappa \\
&+ \frac{252(1+\kappa)(1+3\kappa)(1+4\kappa)(1+12\kappa)(1+14\kappa)}{(1+6\kappa)(1+7\kappa)(1+8\kappa)(2+11\kappa)(2+13\kappa)(3+17\kappa)} P_{0100000}^\kappa \\
P_{0000100}^\kappa \times P_{0000001}^\kappa &= P_{0000101}^\kappa + \frac{4}{1+3\kappa} P_{0001000}^\kappa + \frac{8(1+2\kappa)(1+9\kappa)}{(1+4\kappa)(1+5\kappa)(1+7\kappa)} P_{1000010}^\kappa \\
&+ \frac{90(1+\kappa)(1+3\kappa)(1+11\kappa)}{(1+5\kappa)^2(2+13\kappa)(3+13\kappa)} P_{0100001}^\kappa \\
&+ \frac{40(1+\kappa)(1+2\kappa)(2+7\kappa)(1+10\kappa)(1+11\kappa)}{(1+5\kappa)^3(1+7\kappa)(2+9\kappa)(2+13\kappa)} P_{0010000}^\kappa \\
&+ \frac{96(1+2\kappa)(1+3\kappa)(1+4\kappa)(2+7\kappa)(1+9\kappa)(1+12\kappa)(1+13\kappa)}{(1+5\kappa)(1+6\kappa)(1+7\kappa)^2(1+8\kappa)(2+11\kappa)(2+13\kappa)(3+17\kappa)} P_{0000010}^\kappa \\
P_{0001000}^\kappa \times P_{0000001}^\kappa &= P_{0001001}^\kappa + \frac{5}{1+4\kappa} P_{0110000}^\kappa + \frac{12(1+\kappa)(1+8\kappa)}{(1+3\kappa)(1+5\kappa)(2+7\kappa)} P_{1000100}^\kappa \\
&+ \frac{60(1+\kappa)(1+2\kappa)(1+6\kappa)(1+8\kappa)}{(1+4\kappa)(1+5\kappa)^2(2+7\kappa)(3+11\kappa)} P_{0100010}^\kappa \\
&+ \frac{20(1+\kappa)(1+2\kappa)(2+5\kappa)(1+8\kappa)(1+9\kappa)(3+16\kappa)}{(1+4\kappa)^2(1+5\kappa)^3(3+11\kappa)(4+15\kappa)} P_{0010001}^\kappa \\
&+ \frac{30(1+\kappa)(1+2\kappa)(1+3\kappa)^2(1+8\kappa)(1+10\kappa)(3+17\kappa)}{(1+4\kappa)^2(1+5\kappa)^4(2+9\kappa)(3+13\kappa)} P_{1100000}^\kappa \\
&+ \frac{40(1+\kappa)(1+2\kappa)(1+3\kappa)(2+5\kappa)(1+8\kappa)(1+9\kappa)(1+10\kappa)(3+10\kappa)(2+13\kappa)}{(1+4\kappa)^3(1+5\kappa)^3(2+9\kappa)(2+11\kappa)(3+13\kappa)(4+17\kappa)} P_{0000100}^\kappa \\
P_{0000100}^\kappa \times P_{0000001}^\kappa &= P_{0000101}^\kappa + \frac{4}{1+3\kappa} P_{0001000}^\kappa + \frac{8(1+2\kappa)(1+9\kappa)}{(1+4\kappa)(1+5\kappa)(1+7\kappa)} P_{1000010}^\kappa \\
&+ \frac{90(1+\kappa)(1+3\kappa)(1+11\kappa)}{(1+5\kappa)^2(2+13\kappa)(3+13\kappa)} P_{0100001}^\kappa \\
&+ \frac{40(1+\kappa)(1+2\kappa)(2+7\kappa)(1+10\kappa)(1+11\kappa)}{(1+5\kappa)^3(1+7\kappa)(2+9\kappa)(2+13\kappa)} P_{0010000}^\kappa \\
&+ \frac{96(1+2\kappa)(1+3\kappa)(1+4\kappa)(2+7\kappa)(1+9\kappa)(1+12\kappa)(1+13\kappa)}{(1+5\kappa)(1+6\kappa)(1+7\kappa)^2(1+8\kappa)(2+11\kappa)(2+13\kappa)(3+17\kappa)} P_{0000010}^\kappa \\
P_{0000010}^\kappa \times P_{0000001}^\kappa &= P_{0000011}^\kappa + \frac{3}{1+2\kappa} P_{0000100}^\kappa + \frac{20(1+3\kappa)(1+10\kappa)}{(1+7\kappa)(1+9\kappa)(2+9\kappa)} P_{1000001}^\kappa \\
&+ \frac{42(1+\kappa)(1+4\kappa)(1+14\kappa)}{(1+5\kappa)(1+6\kappa)(1+9\kappa)(2+13\kappa)} P_{0100000}^\kappa \\
&+ \frac{108(1+3\kappa)(1+4\kappa)(1+6\kappa)(1+14\kappa)(1+18\kappa)}{(1+8\kappa)(1+9\kappa)(1+11\kappa)(1+13\kappa)(2+13\kappa)(2+17\kappa)} P_{0000001}^\kappa \\
P_{0000001}^\kappa \times P_{0000001}^\kappa &= P_{0000002}^\kappa + \frac{2}{1+\kappa} P_{0000010}^\kappa + \frac{12(1+4\kappa)}{(1+5\kappa)(1+9\kappa)} P_{1000000}^\kappa \\
&+ \frac{56(1+4\kappa)(1+8\kappa)}{(1+9\kappa)(1+13\kappa)(1+17\kappa)}
\end{aligned}$$

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